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HEAT TRANSFER TO LIQUID METALS FLOWING PAST SPHERES AND ELLIPTICAL-ROD BUNDLES*

CHIA-JUNG HSU

Brookhaven National Laboratory, Upton, New York

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Abstract—Theoretical Nusselt numbers have been derived for the cases of heat transfer to liquid metals flowing past a single sphere, and past an elliptical rod (solitary or one inside a bundle of elliptical rods). The analyses are based on the usual assumptions associated with inviscid potential flow [6, 9].

The normalized hydrodynamic potential drop, $\phi_1/(a + b)$, which appears in the expression for the Nusselt number for elliptical-rod bundles was analytically evaluated using the mathematical function of Howland and McMullen [8] and by applying the principles of conformal transformation. It was found that the numerical values of the corresponding parameter, ϕ_1/D, for flow through circular-rod bundles were applicable here, provided that certain changes in notation were made.

NOMENCLATURE

A_n,	constants;
A_{2n+1}, B_{2n},	coefficients in equation (52);
C,	heat capacity [Btu/lb degF];
D_1,	diameter of a sphere [ft];
E,	the complete elliptic integral of the second kind;
$F(e)$,	as defined by equation (51);
K_1,	a constant;
Nu_D,	over-all Nusselt number, $2bh_D/k$, for an elliptical rod, and $h_{D_1}D_1/k$ for a sphere, dimensionless;
Nu_L,	local Nusselt number, $2bh/k$, dimensionless;
Nu_t,	over-all Nusselt number, $2bh_t/k$ for an elliptical rod, and $\bar{h}_tD_1/k$ for a sphere, dimensionless;
N,	an integer;
P,	pitch [ft];
Pe,	over-all Peclet number, $2b\rho C V/k$ for an elliptical rod, and $\rho C VD_1/k$ for a sphere, dimensionless;
$(Pe)_{V_{max}}$,	over-all Peclet number, $2b\rho C V_{max}/k$, dimensionless;
R_0,	radius of a circle [ft];
Re,	Reynolds number, $2bV\rho/\mu$ for an elliptical rod, and $D_1V\rho/\mu$ for a sphere, dimensionless;
Re_m,	Reynolds number based on V_m, $D_1V_m\rho/\mu$, dimensionless;
T,	temperature [°F];
T_1,	a constant temperature [°F];
T',	temperature excess [degF];
T'_s,	temperature excess on the surface of a sphere [degF];
$\bar{T}'_s$,	average temperature excess on the surface of a sphere [degF];
T'_0,	a constant temperature excess [degF];
T_i,	uniform upstream temperature [°F];
T'_m,	average temperature excess [degF];
V,	uniform upstream fluid velocity [ft/h];
V_m,	mean cross sectional velocity, defined in reference [11] [ft/h];
V_{max},	shell-side fluid velocity across tube bank and based on minimum flow area [ft/h];
Z,	complex function as defined by equation (53);
a, b,	major and minor axis of an ellipse [ft];

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c, $= \sqrt{(a^2 - b^2)}$;
e, eccentricity of an ellipse, c/a;
$\operatorname{erfc} x$, $= 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\delta^2} \, d\delta$;
h, local heat-transfer coefficient in rectangular coordinates [Btu/ft^2 h degF];
$\bar{h}_{D_1}$, average heat-transfer coefficient for a sphere, $= (1/4\pi r_0^2) \iint_s h \, ds$ where s is surface area [Btu/ft^2 h degF];
h', local heat-transfer coefficient in elliptic coordinates [Btu/ft^2 h degF];
h'', local heat-transfer coefficient in ϕ, ψ coordinates [Btu/ft^2 h degF];
$\bar{h}'$, average heat-transfer coefficient in elliptic coordinates [Btu/ft^2 h degF];
$\bar{h}$, average heat-transfer coefficient in rectangular coordinates [Btu/ft^2 h degF];
i, $= \sqrt{-1}$;
k, thermal conductivity [Btu/ft h degF];
n, an integer;
q_{total}, total rate of heat flow from the entire surface of a sphere [Btu/h];
q', rate of heat flow per unit length of elliptic cylinder perpendicular to the direction of flow [Btu/ft h];
q'', surface heat flux [Btu/ft^2 h];
r, radial distance [ft];
r_0, radius of a sphere [ft];
s, arc length of an ellipse in rectangular coordinates [ft];
s_1, arc length of a circle [ft];
s', arc length of an ellipse in elliptic coordinates [ft];
v_r, v_θ, velocity components in r and θ directions;
v_ξ, v_η, velocity components in ξ and η directions;
w, complex potential, $\Phi + i\Psi$;
x, y, z, distance coordinates [ft];

Greek symbols
Γ, gamma function;
Γ_n, as defined by equation (11);
ζ, $\xi + i\eta$, complex function;
η, imaginary part of the complex function ζ, or angle measured from the front stagnation point [degree];
θ, angle measured from the front stagnation point of a circle [degree];
κ, thermal diffusivity [ft^2/h];
λ, $(a + b)/2P$ or $D/2P$;
μ, viscosity of fluid [lb/h ft];
ξ, real part of the complex function;
π, $= 3{\cdot}1416$;
ρ, density of fluid [lb$_m$/ft^3];
Φ, hydrodynamic potential function;
Φ_s, hydrodynamic potential on the surface of an elliptical cylinder;
ϕ_1, unit hydrodynamic potential at the rear stagnation point of an elliptical or circular cylinder;
ϕ, unit hydrodynamic potential, $\Phi/-V$;
ϕ_0, coordinate variable in spherical coordinate;
Ψ, hydrodynamic stream function;
ψ, unit hydrodynamic stream function, $\Psi/-V$.

INTRODUCTION

THEORETICAL Nusselt numbers for heat transfer to liquid metals in cross-flow through circular-rod bundles have been found by the author [9] to agree well with existing experimental results. In that study, the assumption of inviscid potential flow was successfully employed. The purpose of this study was to extend the analysis to include the cases of liquid metal flow past a single sphere and past an elliptical rod (solitary or one inside a bundle of elliptical rods). Simplifying assumptions imposed on the continuity, momentum and energy equations were essentially the same as those used in the previous analysis [9].

It will be shown that Boussinesq's transformation [2], which has been extensively used to analyse heat-transfer problems in potential flow around a circular cylinder, can also be utilized to analyse two-dimensional heat transfer in flow past a sphere for a low Prandtl number fluid. To the author's knowledge, no such attempt has ever been made. For normal cross-flow of liquid metals past an elliptical-rod, it was found that the Nusselt numbers bear a certain relationship to those for the case in which the rods are circular. The average Nusselt number increases as the eccentricity of the ellipse decreases; and, at the limit when the eccentricity becomes zero, the Nusselt numbers reduce to those for the case of normal flow past a single circular rod or through a bundle of circular rods.

The theoretical expression for the parameter, $\phi_1/(a + b)$, which appears in the Nusselt numbers for rod bundles was obtained from the results for circular rods by use of conjugate functions. In a previous paper [9], the corresponding parameter, ϕ_1/D, for circular rods was evaluated analytically, using the mathematical functions of Howland and McMullen [8]. It will be shown that the computational results for ϕ_1/D which were presented in the previous paper are applicable to the evaluation of $\phi_1/(a + b)$ for normal cross-flow through elliptic-rod bundles. It is only necessary to change ϕ_1/D to $\phi_1/(a + b)$, and replace $D/2P$ by $(a + b)/2P$.

FLOW PAST A SINGLE SPHERE

Liquid flow behavior past single spheres is relatively well understood. The transition from a laminar boundary layer to a turbulent one occurs [5] at a critical Reynolds number of ~ 2 to 4×10^5. This transition causes an increase in the angle between the forward stagnation point and the point of separation. For liquid metals, eddy transport of heat does not become significant until Reynolds numbers, considerably above the critical value, are reached. Even in the wake region where eddy motion predominates, the high thermal conductivities of liquid metals can eclipse the effect of eddy heat transport at moderately high Reynolds numbers. This has been observed experimentally for cross-flow through rod bundles by Hoe *et al.* [7] using mercury, and more recently by Borishanskii *et al.* [1] using liquid sodium. Both of these studies showed that the local coefficient decreased to a minimum as the rear stagnation point is approached. For flow of liquid metals past a single sphere or elliptical cylinders, a similar situation undoubtedly prevails, although it has not yet been experimentally confirmed.

Under the assumptions made [9], the energy equation in spherical coordinates can be written as

$$v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} = \frac{k}{\rho C}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right)\right] \quad (1)$$

The coordinate variables are as illustrated in Fig. 1. For flow around a sphere, two dimensional description of both temperature and

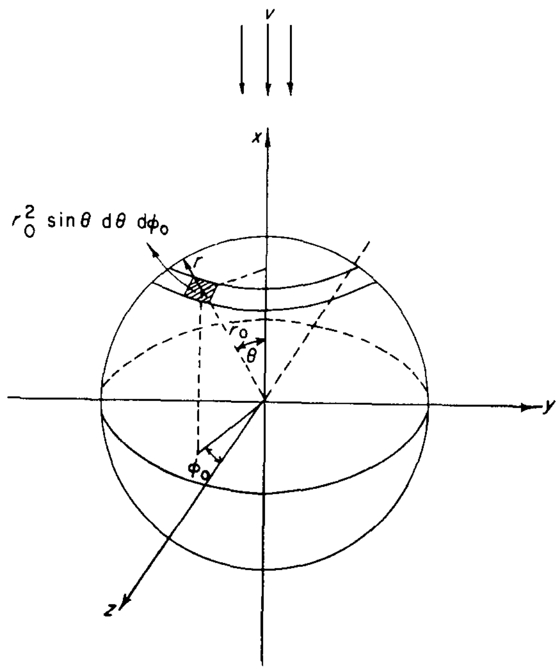

FIG. 1. Coordinate variables in the spherical coordinate.

velocity fields is sufficient since $\partial T/\partial\phi_0 = \partial^2 T/\partial\phi_0^2 = 0$, due to geometrical symmetry. The continuity and momentum equations can thus be replaced by the two-dimensional Laplace equation:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \Psi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Psi}{\partial \theta}\right)=0 \quad (2\text{-}1)$$

or

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right)=0 \quad (2\text{-}2)$$

where the r and θ velocity components are related to Ψ and Φ by

$$v_r=-\frac{1}{r}\frac{\partial \Psi}{\partial \theta}=-\frac{\partial \Phi}{\partial r} \quad (3\text{-}1)$$

and

$$v_\theta=\frac{\partial \Psi}{\partial r}=-\frac{1}{r}\frac{\partial \Phi}{\partial \theta} \quad (3\text{-}2)$$

The form of equations (1) and (2) suggests that Boussinesq's transformation of independent variables [2] is applicable to the solution of this two-dimensional problem. In effect, this transformation causes the circle obtained by projecting a sphere perpendicularly on the x–y plane to be mapped into a line segment. Manipulation of the problem for flow past a sphere differs from that for flow past a cylinder in two ways. First, the two-dimensional stream function and the velocity potential for flow past a sphere differ from those for flow past cylinders. They are given [10] as,

$$\Psi=\tfrac{1}{2}\,Vr^2\sin^2\theta\,[1-(r_0^3/r^3)] \quad (4)$$

and

$$\Phi=V\left(r\cos\theta+\frac{r_0^3\cos\theta}{2r^2}\right) \quad (5)$$

Secondly, it is obvious that a cylinder has an identical circular cross section everywhere when cut by a horizontal plane parallel to the x–y plane. The circumferential temperature distribution obtained by solving the differential equation, resulting from the Boussinesq's transformation, therefore remains unchanged along the surface of the cylinder in the z direction. That this is not the case for the sphere is obvious. Instead, a sphere can be pictured as being formed by revolving the circle about the x-axis (in the ϕ_0 direction). The temperature distribution over the entire surface of the sphere can thus be easily visualized once the temperature along the circumference of the circle is known by solving the differential equation. Thus, applying Boussinesq's transformation of independent variables, neglecting $\partial^2T/\partial\phi^2$ compared to $\partial^2T/\partial\psi^2$, and then changing the temperature variable by letting $T'=T-T_i$, equations (1) and (2) can, as before [9], be simplified to

$$\frac{\partial T'}{\partial \phi}=\frac{\kappa}{V}\frac{\partial^2 T'}{\partial \psi^2} \quad (6)$$

From equation (5), the expression for ϕ can be written as

$$\phi=0{\cdot}75D_1(1-\cos\theta) \quad (5)'$$

Derivation of Nusselt numbers corresponding to cosine-series temperature distribution on the surface of a sphere is illustrated in the following. Results for other cases are summarized in Table 1.

Table 1. Nusselt numbers for flow past a single sphere

Thermal condition on the surface of a sphere	Nusselt number
Uniform surface temperature	$Nu_D=0{\cdot}921\ Pe^{1/2}$
Uniform surface heat flux	$Nu_t=1{\cdot}085\ Pe^{1/2}$ $Nu_D=1{\cdot}128\ Pe^{1/2}$
Cosine surface temperature ($N=1$) $T_s'=\sum_1^N A_n(1-\cos\theta)^n$	$Nu_t=1{\cdot}228\ Pe^{1/2}$ $Nu_D=1{\cdot}843\ Pe^{1/2}$
$T_s'=T_1\exp[K_1(1-\cos\theta)]$ ($K_1=1$)	$Nu_t=1{\cdot}274\ Pe^{1/2}$ $Nu_D=1{\cdot}495\ Pe^{1/2}$

If the temperature excess on the surface of a sphere varies in the θ direction in a manner expressible by the equation

$$T_s'=\sum_{n=1}^{N}A_n(1-\cos\theta)^n \quad (7)$$

then, combining equation (7) and equation (5)′ gives the variation of temperature excess as a function of ϕ, i.e.

$$T_s'=\sum_{n=1}^{N}A_n(\phi/0{\cdot}75D_1)^n \quad (8)$$

The solution of equation (6) corresponding to a surface temperature variation given by equation (8) is [3],

$$T'(\phi) = \sum_{n=1}^{N} A_n(4\phi/0{\cdot}75D_1)^n\, \Gamma(n+1) \times i^{2n}\operatorname{erfc}\left[\frac{\psi}{2}\sqrt{\left(\frac{V}{\kappa\phi}\right)}\right] \quad (9)$$

From this equation, the surface heat flux as a function of ϕ can be obtained as

$$q(\phi) = -k\left(\frac{\partial T'}{\partial\psi}\right)_{\psi=0} = \left(\frac{kCV\rho}{\phi}\right)^{1/2} \times \sum_{n=1}^{N} A_n\Gamma_n(\phi/0{\cdot}75D_1)^n \quad (10)$$

where

$$\Gamma_n = \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \quad (11)$$

The local heat-transfer coefficient, therefore, becomes

$$h(\phi) = \left(\frac{kCV\rho}{\phi}\right)^{1/2} \frac{\sum_{n=1}^{N} A_n\Gamma_n(1-\cos\theta)^n}{\sum_{n=1}^{N} A_n(1-\cos\theta)^n} \quad (12)$$

Since

$$h(\theta) = h(\phi)\frac{\mathrm{d}\phi}{\mathrm{d}s_1} = \sqrt{\left(\frac{3kCV\rho}{D_1}\right)}(1+\cos\theta)^{1/2} \frac{\sum_{n=1}^{N} A_n\Gamma_n(1-\cos\theta)^n}{\sum_{n=1}^{N} A_n(1-\cos\theta)^n} \quad (13)$$

the expression for the local Nusselt number can be written

$$Nu(\theta) = \frac{(\sqrt{3})\,Pe^{1/2}(1+\cos\theta)^{1/2}\sum_{n=1}^{N} A_n\Gamma_n(1-\cos\theta)^n}{\sum_{n=1}^{N} A_n(1-\cos\theta)^n} \quad (14)$$

For a special case when $N = 1$, the distribution of surface temperature excess is from equation (7),

$$T_s'(\theta) = A_1(1-\cos\theta) \quad (15)$$

This particular type of temperature distribution was considered for the case of circular cylinders in the previous analysis [9]. In this case, the local heat-transfer coefficient, from equation (13), becomes

$$h(\theta) = \sqrt{\left(\frac{3kCV\rho}{D_1}\right)}\,\Gamma_1(1+\cos\theta)^{1/2}$$

and the average heat-transfer coefficient can be calculated as

$$\bar{h}_{D_1} = \frac{\Gamma_1}{2}\sqrt{\left(\frac{3kCV\rho}{D_1}\right)}\int_0^{\pi}(1+\cos\theta)^{1/2}\sin\theta\,\mathrm{d}\theta = 1{\cdot}843\left(\frac{kCV\rho}{D_1}\right)^{1/2} \quad (16)$$

Consequently,

$$Nu_{D_1} = \frac{\bar{h}_{D_1}D_1}{k} = 1{\cdot}843\ Pe^{1/2} \quad (17)$$

To obtain the expression for Nu_t, the total heat flux over the entire surface of the sphere is first obtained. Thus,

$$q_{\text{total}} = \sqrt{\left(\frac{kCV\rho}{0{\cdot}75D_1}\right)}\sum_{n=1}^{N} A_n\Gamma_n r_0^2 \int_0^{2\pi}\int_0^{\pi}(1-\cos\theta)^{(2n-1)/2}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi_0 = 4\pi r_0^2\sqrt{\left(\frac{kCV\rho}{0{\cdot}75D_1}\right)}\sum_{n=1}^{N} A_n\Gamma_n\frac{2^{n+\frac{1}{2}}}{(2n+1)} \quad (18)$$

The average temperature excess on the surface of the sphere is

$$\bar{T}_s' = \frac{1}{4\pi r_0^2}\sum_{n=1}^{N} A_n r_0^2 \int_0^{2\pi}\int_0^{\pi}(1-\cos\theta)^n \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi_0 = \frac{1}{2}\sum_{n=1}^{N} A_n\left(\frac{2^{n+1}}{n+1}\right) \quad (19)$$

Hence, the expression for Nu_t finally becomes

$$Nu_t = \frac{D_1 q_{\text{total}}}{4\pi r_0^2 T_s' k} = \frac{1{\cdot}633\, Pe^{1/2} \sum_{n=1}^{N} \left(\frac{2^n}{2n+1}\right) A_n \Gamma_n}{\sum_{n=1}^{N} \left(\frac{2^n}{n+1}\right) A_n} \qquad (20)$$

For the particular case of $N = 1$, equation (20) reduces to

$$Nu_t = 1{\cdot}228\, Pe^{1/2} \qquad (21)$$

Owing to the assumptions made, the equations derived so far may be expected to apply only to fluids having very small Prandtl number, such as liquid metals, and for Reynolds numbers below say 5×10^5. It is not possible, at the present time, to compare these equations with any experimental results, since, to the author's knowledge, they are not available. Recently, Vliet and Leppert [11] reported some experimental results on convection heat transfer to water from an isothermal sphere. Their experimental correlation, based on negligible fluid property variation, is plotted as the dashed line in Fig. 2. The equation from Table 1, for uniform surface temperature, is also plotted in this figure. Although water has a higher Prandtl number compared with liquid metals, certain qualitative conclusions can be drawn from this comparison. At relatively low Reynolds numbers, the equation predicts Nusselt numbers which differ considerably from the experimental measurements for water. This indicates that viscous effects are, in fact, not negligible for ordinary fluids, and it also reveals the well known trend that slug flow analyses generally overpredict heat transfer relative to laminar flow. As the Reynolds number is increased, however, prediction by the equation tends to agree more closely with the experimental results for water. The comparatively rapid increase of the empirical Nusselt number is presumably caused by the fact that the effect of wake flow becomes important at high Reynolds number.

FLOW PAST AN ELLIPTICAL ROD OR THROUGH ELLIPTICAL-ROD BUNDLES

The flow field or temperature field around an elliptical rod is most conveniently described in terms of elliptic cylindrical coordinates. In addition to the assumptions [9] made, it is further assumed that for an elliptical rod located inside the bundle, the distribution of hydrodynamic

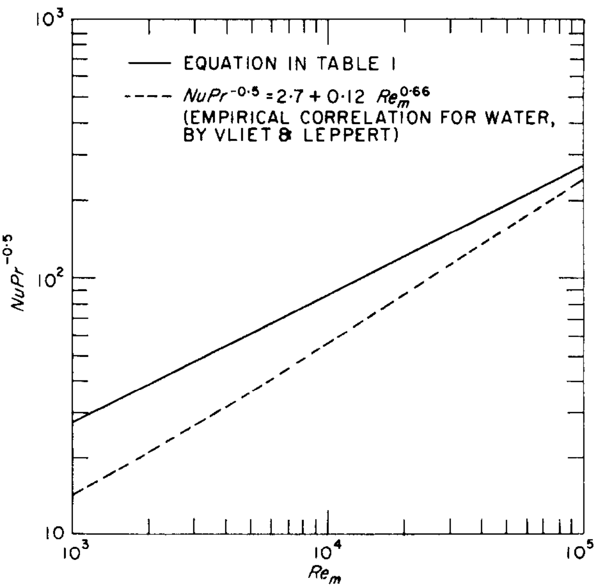

FIG. 2. Plots of Re_m against $NuPr^{-6{\cdot}5}$.

potential around the surface of the rod is of the cosine type in terms of elliptic cylindrical coordinates. Justification of this assumption will be given in a later section. Under these assumptions the energy equation in terms of elliptic cylindrical coordinates can be written as

$$v_\xi \frac{\partial T}{\partial \xi} + v_\eta \frac{\partial T}{\partial \eta} = \frac{\kappa}{a\sqrt{(\sinh^2 \xi + \sin^2 \eta)}} \times \left[\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2}\right]. \quad (22)$$

The equation of continuity and the momentum equation can be combined and replaced by the Laplace equation. If the potential and stream functions in elliptical coordinates are defined by:

$$v_\eta = \frac{1}{a\sqrt{(\sinh^2 \xi + \sin^2 \eta)}} \frac{\partial \Psi}{\partial \xi} = \frac{-1}{a\sqrt{(\sinh^2 \xi + \sin^2 \eta)}} \frac{\partial \Phi}{\partial \eta} \quad (23)$$

and

$$v_\xi = \frac{-1}{a\sqrt{(\sinh^2 \xi + \sin^2 \eta)}} \frac{\partial \Psi}{\partial \eta} = \frac{-1}{a\sqrt{(\sinh^2 \xi + \sin^2 \eta)}} \frac{\partial \Phi}{\partial \xi}; \quad (24)$$

then,

$$\frac{\partial \Psi}{\partial \xi} = -\frac{\partial \Phi}{\partial \eta}, \quad (25)$$

$$\frac{\partial \Psi}{\partial \eta} = \frac{\partial \Phi}{\partial \xi} \quad (26)$$

and therefore the two-dimensional Laplace's equation retains the form:

$$\frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2} = 0 \quad (27)$$

or

$$\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} = 0 \quad (28)$$

It is not difficult to show that Boussinesq's transformation (1) is also valid for elliptical coordinates. Thus, equation (6) remains valid for this case.

In the following, a derivation of Nusselt number corresponding to the case of a constant temperature at the wall of an elliptical-rod (for flow past both a solitary elliptical-rod and one inside a bundle of elliptical rods) is presented. Results for other cases may be found in Table 2.

Table 2. Nusselt numbers for flow past an elliptical-rod

Thermal condition on the surface of an elliptical-rod	*Nu* number for flow past a solitary elliptical-rod	*Nu* number for flow past an elliptical-rod inside a bundle of elliptical-rods
Uniform surface temperature	$Nu_D = 1{\cdot}128\, F(e) Pe^{1/2}$ where $F(e)$ is given by equation (51)	$Nu_D = 0{\cdot}798 \left(\frac{\phi_1}{a+b}\right)^{1/2} F(e) Pe^{1/2}$
Uniform surface heat flux	$Nu_t = 1{\cdot}272\, F(e) Pr^{1/2}$	$Nu_t = 0{\cdot}90 \left(\frac{\phi_1}{a+b}\right)^{1/2} F(e) Pe^{1/2}$
	$Nu_D = 1{\cdot}489\, F(e) Pe^{1/2}$	$Nu_D = 1{\cdot}051 \left(\frac{\phi_1}{a+b}\right)^{1/2} F(e) Pe^{1/2}$
Cosine surface temperature $T_s' = A_1(1 - \cos\theta)$	$Nu_t = 1{\cdot}505\, F(e) Pe^{1/2}$	$Nu_t = 1{\cdot}064 \left(\frac{\phi_1}{a+b}\right)^{1/2} F(e) Pe^{1/2}$
	$Nu_D = 2{\cdot}258\, F(e) Pe^{1/2}$	$Nu_D = 1{\cdot}597 \left(\frac{\phi_1}{a+b}\right)^{1/2} F(e) Pe^{1/2}$

a. *Solitary elliptical rod*

The solution to equation (6) which corresponds to a constant temperature, T_0', on the surface of the elliptical rod is given [3] as,

$$q''(\phi) = (\rho C V k / \pi \phi)^{1/2} T_0' \qquad (29)$$

hence,

$$h''(\phi) = (\rho C V k / \pi \phi)^{1/2} \qquad (30)$$

Since the above heat-transfer coefficient, $h''(\phi)$, is based upon a unit increment ϕ on the surface of a flat plate, it is necessary to convert it to one based upon a unit increment on the surface of the ellipse. Let $h'(\eta)$ be the heat-transfer coefficient based upon a unit area on the surface of ellipse in elliptical coordinates, then

$$h'(\eta) = h''(\phi) \frac{d\phi}{ds'} \qquad (31)$$

The complex potential for flow around a single elliptical cylinder is given in terms of elliptical coordinates as

$$w = \Phi + i\Psi = V(a+b) \cosh (\zeta - \xi_0) \qquad (32)$$

where

$$\zeta = \xi + i\eta.$$

From equation (32), the potential function can be readily found as

$$\Phi = V(a+b) \cosh (\xi - \xi_0) \cos \eta$$

On the surface of an ellipse, $\xi = \xi_0$, therefore, the distribution of hydrodynamic potential is given by

$$\Phi = V(a+b) \cos \eta \qquad (33)$$

For the present purpose, the potential function around a single elliptical cylinder is written as

$$\phi = (a+b)(1 - \cos \eta) \qquad (34)$$

The arc length, s', along the ellipse, $\xi = \xi_0$, can be expressed in elliptical coordinates as $s' = \xi_0 \eta$. Therefore, $d\phi/ds' = (a+b) \sin \eta / \xi_0$, and from equation (31),

$$h'(\eta) = \frac{\sqrt{(a+b)}}{\xi_0} \sqrt{\left(\frac{\rho C V k}{\pi}\right)} (1 + \cos \eta)^{1/2} \qquad (35)$$

In order to obtain the expression for the Nusselt number, it is necessary to convert h' to rectangular coordinates. The two types of coordinate systems are related by

$$x = c \cosh \xi \cos \eta \qquad (36)$$

and

$$y = c \sinh \xi \sin \eta \qquad (37)$$

On the surface of an ellipse, $\xi = \xi_0$, therefore,

$$x = c \cosh \xi_0 \cos \eta = a \cos \eta$$

$$y = c \sinh \xi_0 \sin \eta = b \sin \eta$$

The incremental distance, ds, on the surface of an ellipse in rectangular coordinates is

$$ds = \sqrt{[(dx)^2 + (dy)^2]} = \sqrt{(a^2 \sin^2 \eta + b^2 \cos^2 \eta)}\, d\eta \qquad (38)$$

Let ds' be the incremental distance on the surface of the ellipse in elliptical coordinates, then

$$h(\eta) = h'(\eta) \frac{ds'}{ds} = h'(\eta) \xi_0 \frac{d\eta}{ds} \qquad (39)$$

Hence, combining equations (35), (38), and (39) gives

$$h(\eta) = \sqrt{(a+b)} \sqrt{\left(\frac{\rho C V k}{\pi}\right)} \left(\frac{1 + \cos \eta}{a^2 \sin^2 \eta + b^2 \cos^2 \eta}\right)^{1/2} \qquad (40)$$

Accordingly, the expression for the local Nusselt number, Nu_L, can be written as

$$Nu_L(\eta) = \frac{2bh}{k} = \sqrt{\left(\frac{2b(a+b)}{\pi}\right)} (Pe)^{1/2} \left(\frac{1 + \cos \eta}{a^2 \sin^2 \eta + b^2 \cos^2 \eta}\right)^{1/2} \qquad (41)$$

If a and b, the major and minor axes of an ellipse, are written in terms of the eccentricity, e, then, since $e = c/a = \sqrt{(a^2 - b^2)}/a$, equation (41) can be written in equivalent form:

$$Nu_L(\eta) = \sqrt{(2/\pi)} \sqrt{\left(\frac{(1 - e^2) + \sqrt{(1 - e^2)}}{1 - e^2 \cos^2 \eta}\right)} (Pe)^{1/2} (1 + \cos \eta)^{1/2} \qquad (42)$$

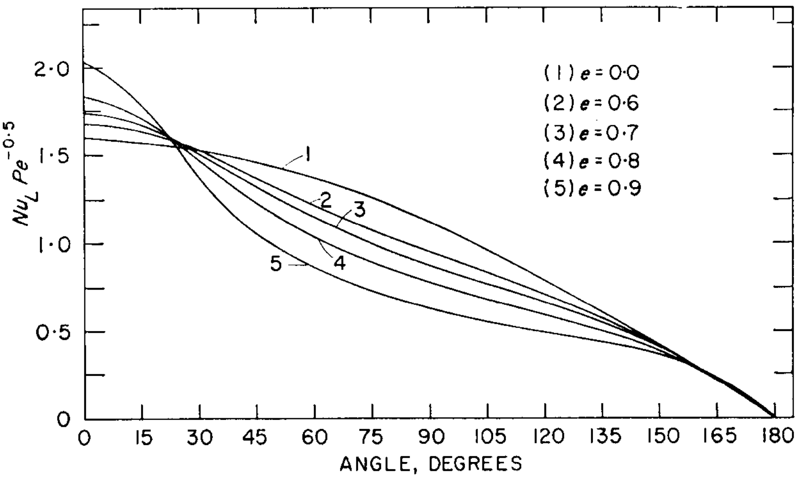

FIG. 3. Local Nusselt number as a function of η, the angle measured from the forward stagnation point.

In Fig. 3 the local Nusselt number, expressed as $Nu_L Pe^{-1/2}$, is plotted against the angle measured from the forward stagnation point. It can be observed from the plots that as the eccentricity increases and approaches unity, the local Nusselt number starts to decrease more and more rapidly in the front region ($\eta = 0$ to $\sim 60°$) of the ellipse. Drake *et al.* [4] have experimentally measured local Nusselt numbers for flow of air past an elliptical-rod having eccentricity of 0·943 (a:b = 3:1). Their results also show the trend of the rapid decrease of local Nusselt number in the front region of the elliptical cylinder.

To obtain the average Nusselt number, Nu_D, the average heat-transfer coefficient in elliptic cylindrical coordinates is first obtained. Thus:

$$\bar{h}' = \frac{\sqrt{(a+b)}}{\pi\xi_0}\sqrt{\left(\frac{CVk\rho}{\pi}\right)}\int_0^{\pi}(1+\cos\eta)^{1/2}\,d\eta$$

$$= \frac{2\sqrt{[2(a+b)]}}{\pi\xi_0}\sqrt{\left(\frac{CVk\rho}{\pi}\right)} \quad (43)$$

To convert the above expression to one in rectangular coordinates, it is noted that

$$\pi\xi_0\bar{h}' = 2aE(K)\bar{h}$$

where $K = \sqrt{[1-(b/a)^2]}$ and $E(K)$ is the complete elliptic integral of the second kind. Accordingly,

$$\bar{h} = \frac{\sqrt{[2(a+b)]}}{aE(K)}(\rho CVk/\pi)^{1/2} \quad (44)$$

and the average Nusselt number, Nu_D, becomes

$$Nu_D = 1\cdot128(Pe)^{1/2}\frac{(1+b/a)^{1/2}(b/a)^{1/2}}{E(K)} \quad (45)$$

If $a = b$, the above equation reduces to

$$Nu_D = 1\cdot015(Pe)^{1/2} \quad (46)$$

which is the Nusselt number for flow past a circular cylinder [6]; and, if $b = 0$, it reduces to

$$Nu_D = 1\cdot128(Pe)^{1/2} \quad (46)'$$

which is the Nusselt number for flow past a flat plate [6]. In the latter case, the characteristic length inside the expression of Nu_D and Pe has been changed to the length of the flat plate.

If eccentricity, e, is used, equation (45) can also be written in an alternative form:

$$Nu_D = 1\cdot128(Pe)^{1/2}\frac{\sqrt{[1-e^2)+\sqrt{(1-e^2)}]}}{E(e)} \quad (47)$$

A circle is an ellipse of zero eccentricity. It is apparent that equation (47) reduces to equation (46) if e is zero.

b. *An elliptical rod located in the interior of an elliptical-rod bundle*

The foregoing analysis can be readily extended to obtain the expression for Nusselt numbers for the case of flow past an elliptical-rod located inside a bundle of elliptical-rods. This is illustrated in the following.

From equation (29), the rate of heat flow from the entire surface of the elliptical rod can be obtained as:

$$q' = 2\int_0^{\phi_1} q''(\phi)\,d\phi = 4T_0'\left(\frac{CV\rho k}{\pi}\right)^{1/2}\phi_1^{1/2}$$

where ϕ_1 represents the value of ϕ at the rear stagnation point of an elliptical-rod. Analytical evaluation of ϕ_1 will be discussed in the next section. From the above equation, the heat-transfer coefficient can be written as

$$h_D = \frac{q'}{4aE(K)T_0'} = \frac{1}{\sqrt{\pi}}\,\frac{\phi_1^{1/2}}{aE(K)}(\rho CVk)^{1/2} \quad (48)$$

and, consequently, the Nusselt number, Nu_D, becomes:

$$Nu_D = 0{\cdot}798\left(\frac{\phi_1}{a+b}\right)^{1/2} Pe^{1/2}\,\frac{(1+b/a)^{1/2}(b/a)^{1/2}}{E(K)} \quad (49)$$

If the Peclet number is based upon the velocity of fluid flowing through the minimum flowing area, equation (49) can also be written as:

$$Nu_D = 0{\cdot}798\left(\frac{\phi_1}{a+b}\right)^{1/2} F(e)(Pe)^{1/2}_{V_{\max}}\,(V/V_{\max})^{1/2} \quad (50)$$

where

$$F(e) = \frac{\sqrt{[(1-e^2)+\sqrt{(1-e^2)}]}}{E(e)} \quad (51)$$

Once again, it is not possible at the present time to compare the theoretical results presented in this paper with any experimental data, due to the lack of latter information. Confirmation of the results of this analysis has to rely upon future experimental work.

c. *Theoretical evaluation of the parameter,* $\phi_1/(a+b)$

In the expressions for the Nusselt number presented in the previous sections, a term, $\phi_1/(a+b)$, appears in each of the equations for rod bundles. This term represents the difference in hydrodynamic potential between the front and rear stagnation points of an elliptical rod located inside a bundle, in terms of elliptical coordinates. The analytical evaluation of this parameter will now be given by applying the principles of conformal transformation and by using conjugate functions.

In the previous paper [9], it was shown that the potential field around a circular cylinder located in the interior of a bundle could be calculated analytically, using the mathematical functions of Howland and McMullen [8]. The potential field around the circular rod was found to be given by the expression:

$$\Phi = VR_0\left(\sum_{n=0}^{\infty}\{A_{2n+1}\lambda^{-2n}[(R_0/r)^{2n+1} + (r/R_0)^{2n+1}]\sin(2n+1)\theta\} - \sum_{n=1}^{\infty}\{B_{2n}\lambda^{-2n+1}[(R_0/r)^{2n} + (r/R_0)^{2n}]\cos 2n\theta\}\right) \quad (52)$$

where λ is $D/2P$, and the constants, A_{2n+1} and B_{2n}, are as given in the previous paper.

To calculate the potential field in terms of elliptical coordinates around an elliptical rod located inside a bundle, the following consecu-secutive conformal transformations are made:

$$Z = \tfrac{1}{2}[z + \sqrt{(z^2 - c^2)}] \quad (53)$$

and

$$z = c\cosh\zeta = c\cosh(\xi + i\eta) \quad (54)$$

The first transformation maps the group of circular cylinders defined by Howland and McMullen [8] into a group of elliptical cylinders. The second transformation changes the variables from rectangular coordinates to elliptical cylindrical coordinates [10]. From equation (54) it can be readily seen that:

$$\sqrt{(z^2 - c^2)} = c\sinh\zeta \quad (55)$$

and accordingly:

$$z + \sqrt{(z^2 - c^2)} = c(\sinh\zeta + \cosh\zeta) = ce^{\zeta} \quad (56)$$

$$z - \sqrt{(z^2 - c^2)} = c(\cosh\zeta - \sinh\zeta) = ce^{-\zeta} \quad (57)$$

On an ellipse, $\xi = \xi_0$, the major and minor axes of the ellipse, a and b, respectively, can be expressed by

$$a = c \cosh \xi_0, \; b = c \sinh \xi_0 \quad (58)$$

Therefore:

$$a + b = c(\cosh \xi_0 + \sinh \xi_0) = ce^{\xi_0} \quad (59)$$

$$a - b = c(\cosh \xi_0 - \sinh \xi_0) = ce^{-\xi_0} \quad (60)$$

The relationship between the original variables and the final transformed variables can be found by noting that $Z = re^{i\theta}$, and then combining this expression with equations (53) and (56). Thus:

$$Z = re^{i\theta} = \tfrac{1}{2}\,[z + \sqrt{(z^2 - c^2)}] = \tfrac{1}{2}\,ce^{\zeta} = (ce^{\xi}/2)e^{i\eta} \quad (61)$$

It is seen, therefore, that the two consecutive transformations cause the following transformations of variables:

$$r \to ce^{\xi}/2 \quad \theta \to \eta \quad (62)$$

It is also noted that the circles, $r = R_0$, in the original rectangular coordinates are mapped into ellipses, $\xi = \xi_0$. Therefore, from equations (62) and (59), one can write:

$$R_0 \to ce^{\xi_0}/2 = (a + b)/2 \quad (63)$$

Since the solution to the Laplace equation,

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0, \quad (64)$$

with the appropriate boundary conditions, is known to be [8, 9]:

$$\Psi = VR_0 \left(\sum_{n=0}^{\infty} \{A_{2n+1}\lambda^{-2n}\,[(R_0/r)^{2n+1} - (r/R_0)^{2n+1}] \cos (2n + 1)\theta\} + \sum_{n=1}^{\infty} \{B_{2n}\lambda^{-2n+1}\,[(R_0/r)^{2n} - (r/R_0)^{2n}] \sin 2n\theta\} \right) \quad (65)$$

the stream function for the flow around the elliptical rods in elliptical coordinates can be written as:

$$\Psi = V(a + b) \left\{ \sum_{n=0}^{\infty} [A_{2n+1}\lambda^{-2n} \sinh (2n + 1)(\xi_0 - \xi) \cos (2n + 1)\eta] + \sum_{n=1}^{\infty} [B_{2n}\lambda^{-2n+1} \sinh 2n(\xi_0 - \xi) \sin 2n\eta] \right\} \quad (66)$$

where λ is now $(a + b)/2P$. From equation (66), it can be readily seen that on the surfaces of the ellipses, $\xi = \xi_0$, $\Psi = 0$.

Similarly, the potential function for the flow around the elliptical rods can be found in terms of elliptical coordinates as:

$$\Phi = V(a + b) \left\{ \sum_{n=0}^{\infty} [A_{2n+1}\lambda^{-2n} \cosh (2n + 1)(\xi - \xi_0) \sin (2n + 1)\eta] - \sum_{n=1}^{\infty} [B_{2n}\lambda^{-2n+1} \cosh 2n(\xi - \xi_0) \cos 2n\eta] \right\} \quad (67)$$

The distribution of hydrodynamic potential on the surface of the elliptical rods can be obtained by letting $\xi = \xi_0$ in equation (67). Thus:

$$\Phi_s = V(a + b) \left\{ \sum_{n=0}^{\infty} [A_{2n+1}\lambda^{-2n} \sin (2n + 1)\eta] - \sum_{n=1}^{\infty} [B_{2n}\lambda^{-2n+1} \cos 2n\eta] \right\} \quad (68)$$

and accordingly the difference in hydrodynamic potential between the forward and rear stagnation points of the elliptical cylinder can be obtained by forming the difference of Φ_s at $\eta = \pi/2$ and $\eta = 3\pi/2$. Finally, this can be written as:

$$\phi_1/(a + b) = 2 \sum_{n=0}^{\infty} (-1)^n A_{2n+1}\lambda^{-2n} \quad (69)$$

From the above equation, it can be seen that the final mathematical form for the parameter is identical to that for flow around circular cylinders. The only differences are that the term ϕ_1/D is replaced by $\phi_1/(a + b)$, and λ now stands for $(a + b)/2P$.

In the previous paper [9], theoretical values of ϕ_1/D were presented as a function of λ. These values are, therefore, still useful for finding $\phi_1/(a + b)$, provided that the above changes are

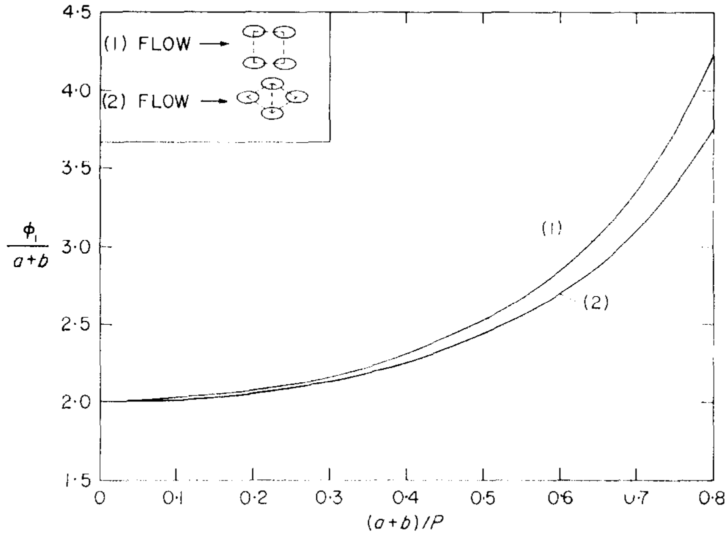

FIG. 4. Plots of normalized hydrodynamic potential drop, $\phi_1/(a+b)$, vs $(a+b)/P$.

made. The plots of $\phi_1/(a+b)$ against $(a+b)/P$ are shown in Fig. 4, for two different tube-bank geometries, i.e. the centers of the ellipses are arranged in square and equilateral triangular fashions.

In deriving the Nusselt numbers for rod bundles, it was assumed that the distribution of hydrodynamic potential along the surface of an elliptical rod located in the interior of rod bundles can be expressed by a cosine function, in terms of elliptical coordinates. This assumption can be analytically justified by using equation (68). In the previous paper [9], similar justification was shown for flow through circular-rod bundles. Since equation (68) has an identical form to the corresponding equation for flow around circular rods, Figs. 9 and 10 in the aforementioned reference [9] can now be interpreted as the plots of $\phi_1/(a+b)$ against the angle measured from the forward stagnation point. It is thus seen that the said assumption is reasonable.

SUMMARY

(1) Analytical expressions for Nusselt numbers for liquid metals flowing past a single sphere, and past an elliptical rod (solitary or one inside a bundle) were obtained by assuming inviscid flow. For the elliptical rods, the Nusselt numbers were expressed as a function of the eccentricity of the ellipse. It was found that the Nusselt numbers bear constant relationship to those for cross-flow through circular-rod bundles, i.e.

For a single rod:

$$Nu_{\text{ell}}(Pe) = Nu_{\text{cyl}}(Pe) \frac{E(0)}{E(e)} \sqrt{\left[\frac{(1-e^2)+\sqrt{(1-e^2)}}{2}\right]}$$

For a rod inside a bundle:

$$Nu_{\text{ell}}[Pe, \phi_1/(a+b)] = Nu_{\text{cyl}}(Pe, \phi_1/D) \frac{E(0)}{E(e)} \sqrt{\left[\frac{(1-e^2)+\sqrt{(1-e^2)}}{2}\right]}$$

(2) The parameter, $\phi_1/(a+b)$, which represents the hydrodynamic potential drop in terms of elliptical coordinates is analytically obtained by using conjugate functions. It was found that the numerical results of ϕ_1/D presented in the previous paper are still applicable, provided that ϕ_1/D is changed to $\phi_1/(a+b)$ and $D/2P$ to $(a+b)/2P$.

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Résumé—Les nombres de Nusselt théoriques ont été obtenus dans les cas du transport de chaleur dans des métaux liquides s'écoulant le long d'une sphère isolée, et le long d'une barre elliptique (isolée ou à l'intérieur d'un faisceau de barres elliptiques). Les analyses sont basées sur les hypothèses habituelles associées avec l'écoulement potential non-visqueux [6, 9].

La chute de potentiel hydrodynamique normalisée, $\phi_1/(a + b)$ qui apparaît dans l'expression du nombre de Nusselt pour des faisceaux de barres elliptiques a été évaluée analytiquement à l'aide de la fonction mathématique de Howland et McMullen [8] et en appliquant les principes de la transformation conforme. On a trouvé que les valeurs numériques du paramètre correspondant, ϕ_1/D, pour l'écoulement à travers des faisceaux de barres circulaires, s'appliquaient ici, pourvu que certains changements de notations soient faits.

Zusammenfassung—Für den Wärmeübergang an flüssige Metalle, die an einer einzigen Kugel und einem elliptischen Stab (Einzelstab oder Stab innerhalb eines Bündels von elliptischen Stäben) vorbeiströmen, wurden theoretische Nusseltzahlen abgeleitet. Die Analysen beruhen auf den üblichen Annahmen über eine nicht-zähigkeitsbehaftete Potentialströmung [6, 9].

Der normalisierte hydrodynamische Potentialabfall $\phi_1/(a + b)$, der in dem Ausdruck der Nusseltzahl für das Bündel elliptischer Stäbe auftritt, wurde analytisch unter Verwendung der mathematischen Funktion von Howland und McMullen [8] und der Theorien der konformen Abbildung geschätzt. Es ergab sich, dass die numerischen Werte der entsprechenden Parameter ϕ_1/D für die Strömung durch ein Bündel von Rundstäben hier angewandt werden durften, vorausgesetzt, dass gewisse Änderungen in den Bezeichnungen gemacht wurden.

Аннотация—Теоретическим путем получены числа Нуссельта для случаев теплообмена к жидким металлам, обтекающим отдельный шар и эллиптический стержень (самостоятельный или внутри пучка эллиптических стержней). Анализы основаны на обычных предположениях, связанных с потенциальными течениями.

Нормализованное падение гидродинамического потенциала $\phi_1/(a + b)$, которое встречается в выражении для числа Нуссельта, описывающем пучки эллиптических стержней, вычислено с помощью математической функции Хауленда и Макмаллена с применением принципов конформных отображений. Найдено, что численные значения соответствующего параметра ϕ_1/D для течения через пучок круглых стержней применимы и в данном случае при условии, что внесены определенные изменения в обозначения.